Math 53, Discussion 107 (Spherical Coordinates and Jacobian)

Wednesday, April 1, 2020 9:17 AM

Definition:

To convert to spherical coordinates, we let

$x = \rho \sin\varphi \cos\theta$

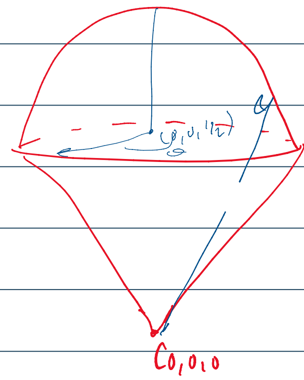
$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

And we need to add the integrating factor: $\rho^2 \sin\varphi$

https://mathinsight.org/spherical_coordinates

Ex: Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below $z = x^2 + y^2 + z^2$



$0 = x^2 + y^2 + z^2 - z$

$0 = x^2 + y^2 + (z - \frac{1}{2})^2 - \frac{1}{4}$

$\rho\cos\varphi = \sqrt{\rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta}$

$= \rho \sin\varphi$

$\Leftrightarrow \cos\varphi = \sin\varphi \Leftrightarrow \varphi = \pi/4 \Leftrightarrow \{\text{all } \varphi, \theta, \rho \text{ s.t. } \varphi = \pi/4\}$

$\rho\cos\varphi = \rho^2 \Leftrightarrow \cos\varphi = \rho$

$R = \{\text{all } \varphi, \theta, \rho \text{ s.t. } 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi/4,\ 0 \le \rho \le \cos\varphi\}$

15.9: Change of coordinates (Jacobian)

Ex. polar $y = r\sin\theta$, $x = r\cos\theta$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

In two dimensions:

$$\iint_R f(x,y)\,dA = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2+y^2 \, dx\,dy = \int_0^1 \int_0^{2\pi} r^2 \; r\,d\theta\,dr$$

$$\int_0^1 (2x)^2\,dx = \int_0^1 4x^2 = \frac{4x^3}{3}\Big|_0^1 = \frac{4}{3}$$

let $u = 2x$ $\quad \frac{du}{dx} = 2$ $\quad dx = \frac{du}{2}$

$$\int_0^2 u^2 \frac{1}{2}\,du = \frac{1}{2}\frac{u^3}{3}\Big|_0^2 = \frac{4}{3}$$

$$\int_0^1 x\,dx = \frac{1}{2}$$

$$\int_0^1 (2x)^2\,dx = \int u^2 \left|\frac{\partial x}{\cancel{\partial u}}\right| \cancel{du}$$

$$\int_0^1 x\,dx = \frac{1}{2}$$

let $u = -x$

$$\int_0^{-1} (-u)\,|(-1)|\,du = \int_0^{-1} -u\,du = \int_{-1}^{0} u\,du = \frac{u^2}{2}\Big|_{-1}^{0} = -\frac{1}{2}$$

$$\int_0^1 (2x)^2\,dx = \int_S u^2 \underbrace{\left|\frac{\partial x}{\partial u}\right|}_{\det\left(\frac{\partial x}{\partial u}\right)} du$$

$$\int_{T(\Omega)} f(\vec{x})\,dV(\vec{x}) = \int_\Omega f(T(\vec{x}))\,|\underbrace{\det(D\,T)(\vec{x})}|\,dV(\vec{x})$$

$T: \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \cdots & \\ \frac{\partial T_2}{\partial x_1} & \cdots & \\ \vdots & & \end{pmatrix}$$

Note that in one variable, this is just normal u substitution

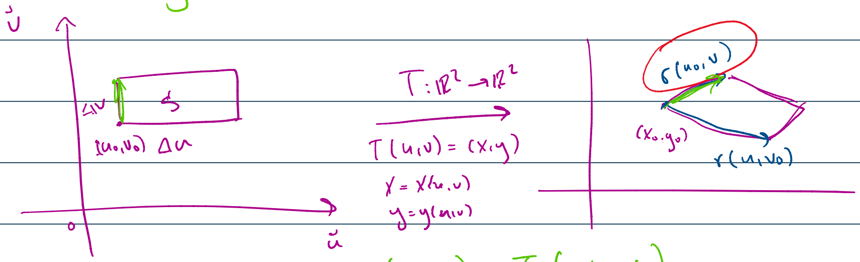
Ex: set up the change of coordinates for $x = r\cos\theta$, $y = r\sin\theta$

Exercises:

1. Describe what the surface $\phi = \frac{\pi}{3}$ looks like
2. Evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV$ where E is enclosed by the sphere $x^2+y^2+z^2 = 9$ and the first octant
3. Set up the determinant to determine the integration constant for integrating in spherical coordinates
4. Show that a standard 1 variable u substitution follows from the change of variables formula
5. Find the Jacobian of the transformation $x = uv$, $y = \frac{u}{v}$
6. Compute $\iint_R x^2\,dA$ where R is the region bounded

by the ellipse $9x^2 + 4y^2 = 36$, (use the sub $x = 2u, y = 3v$)

Prove change of variables in 2D



$r(u_0,v) = T(u_0,v)$

$r(u,v)$ $= g(u,v)\,i + h(u,v)\,j$

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